When a picture is a proof

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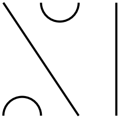
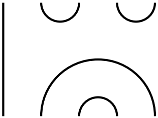
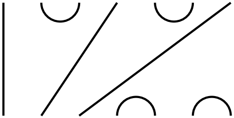
Illustrating Number Theory and Algebra
ICERM, 25 October 2019

the Temperley-Lieb algebra

Definition

A Temperley-Lieb diagram is a non-crossing pairing of n points above and n points below.

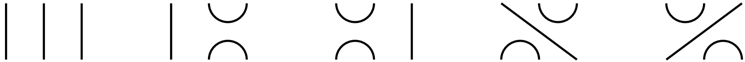
EG:  or  or 

(Two diagrams that are topologically the same, are the same)

Question

How many Temperley-Lieb diagrams on 2n points are there?

When $n = 1$ there is one such diagram; when $n = 2$ there are two; when $n = 3$, five:



Question

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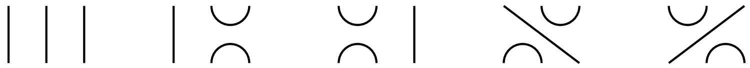
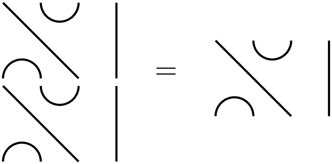


The number of TL_n diagrams is counted by the Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$.

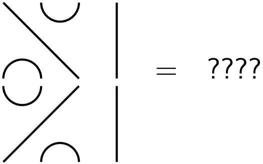
Exercise

Find a bijection between TL_n diagrams and allowed arrangements of $2n$ parentheses.

We can multiply Temperley-Lieb diagrams!
EG:



Okay, but:



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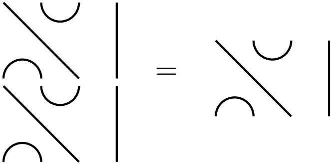

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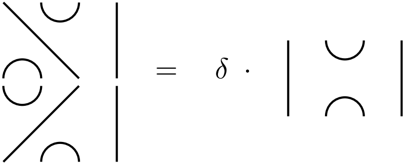

Is it associative?

Definition

The Temperley-Lieb algebra TL_n *for* $n \geq 0$*:*

- *As a vector space (over* $\mathbb{C}[\delta]$*), its basis is Temperley-Lieb diagrams on* $2n$ *points;*
- *Addition is formal;*
- *Multiplication is the linear extension of multiplication-by-stacking.*

What is TL_0?

Definition

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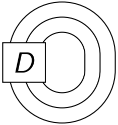
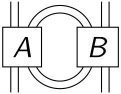
- *Addition is formal;*
- *Multiplication is the linear extension of multiplication-by-stacking.*

What is TL_0? $TL_0 \simeq \mathbb{C}[\delta]$.

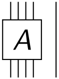
This makes the "capping" map from TL_{2n} to TL_0 into a trace:

$D \mapsto$ 

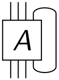
Temperley-Lieb has lots of additional structure:

Funny multiplications, EG: $(A, B) \mapsto$ 

We also have inclusions $TL_n \hookrightarrow TL_{n+1}$ given by

$A \mapsto$ 

And conditional expectations $TL_{n+1} \to TL_n$ given by

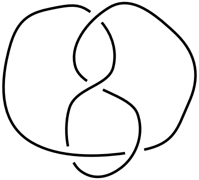
$A \mapsto$ 

This additional structure is encompassed by saying that Temperley-Lieb is a <u>planar algebra</u>.

Knots and knot diagrams

Definition

A knot is the image of a smooth embedding $S^1 \to \mathbb{R}^3$.



Question: Are knots one-dimensional, or three?

Knots and knot diagrams

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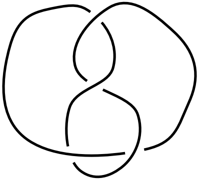
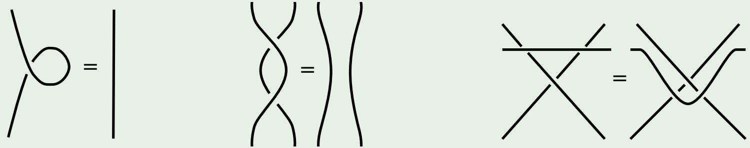
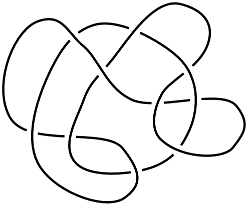
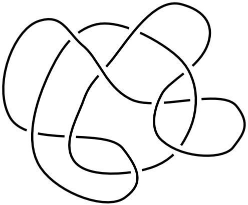
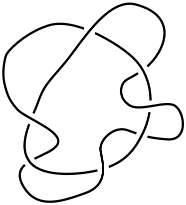


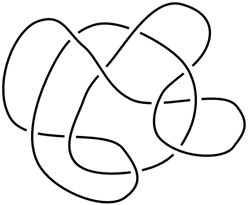
Question: Are knots one-dimensional, or three?
Answer: No.

Theorem (Reidemeister)

If two diagrams represent the same knot, then you can move between them in a series of Reidemeister moves:



 ,  , 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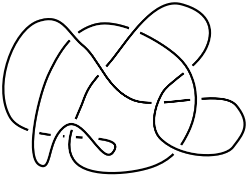
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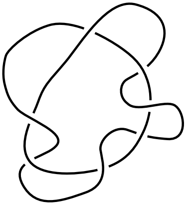

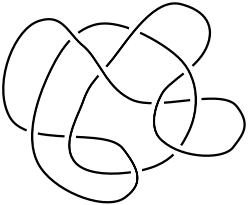
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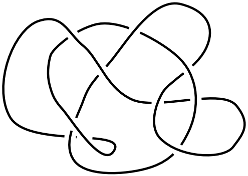
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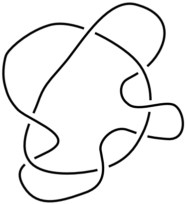

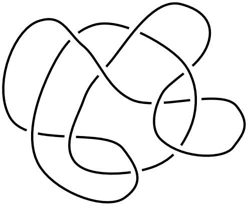

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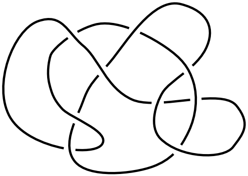

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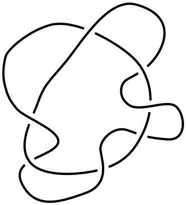
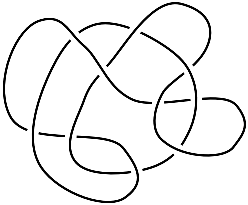
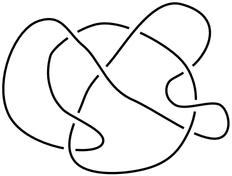
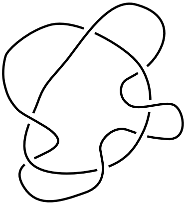
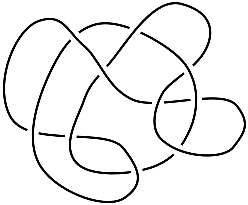


 ,  , 



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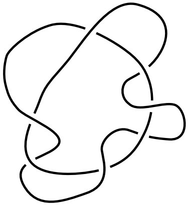

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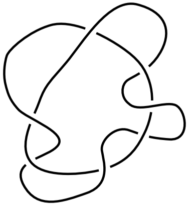
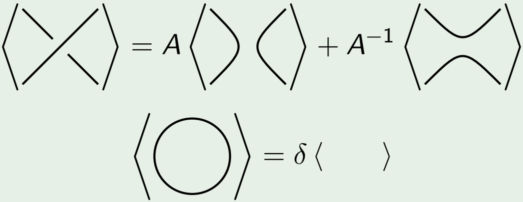
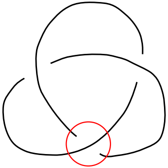
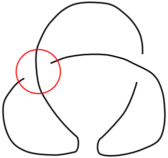
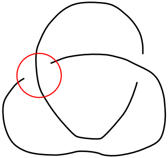


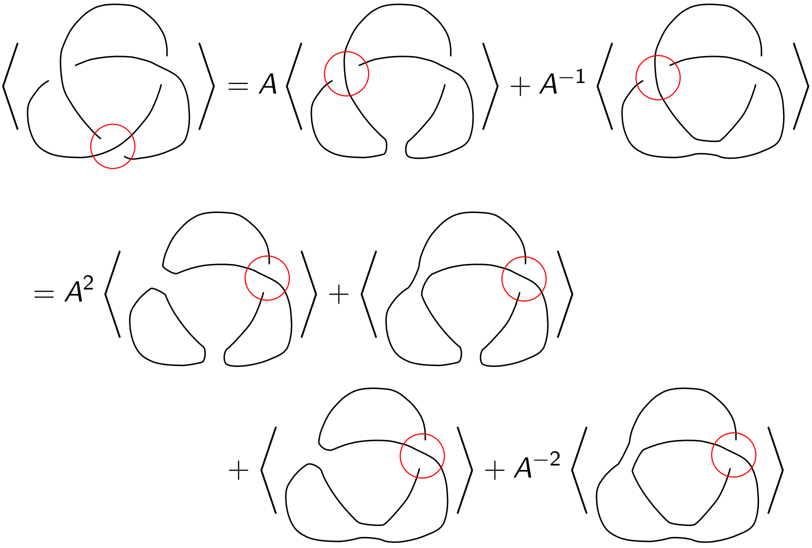
A knot invariant is a map from knot diagrams to something simpler: say, $\mathbb{C}$, or polynomials, or 'simpler' diagrams. Crucially, the value of the invariant shouldn't change under Reidemeister moves.

Definition

The Kauffman bracket is a map from tangles (knots with loose ends) to TL. Let A satisfy $\delta = -A^2 - A^{-2}$. Then define



$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$


$= A$
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$= A^2$
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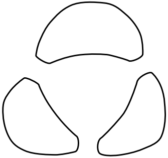
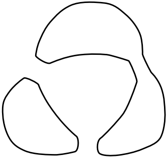
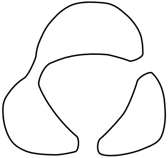
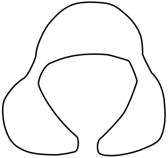
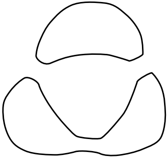
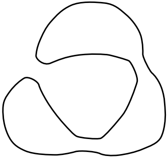
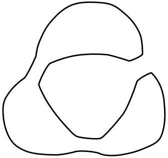
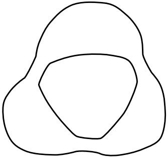
$+ A^{-2}$

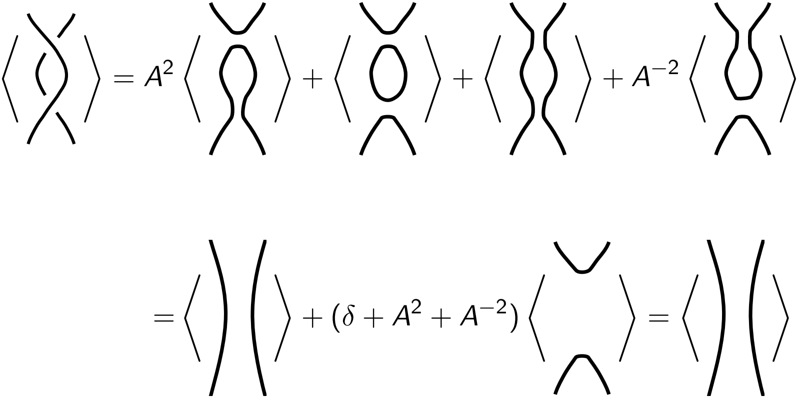
$= A^3 \Big\langle$  $\Big\rangle + A \Big\langle$  $\Big\rangle + A \Big\langle$  $\Big\rangle$

$+ A^{-1} \Big\langle$  $\Big\rangle + A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$+ A^{-1} \Big\langle$  $\Big\rangle + A^{-3} \Big\langle$  $\Big\rangle$

$$= A^3\delta^3 + A\delta^2 + \cdots = -A^9 + A + A^{-3} + A^{-7}$$

The Kauffman bracket is invariant under Reidemeister 2:



Exercise

The Kauffman bracket is also invariant under Reidemeister 3, but it is not invariant under Reidemeister 1.

A modification of the Kauffman bracket which is invariant under Reidemeister 1 is known as the Jones Polynomial when applied to knots.

The Jones polynomial is pretty good, but not perfect, at telling knots apart.

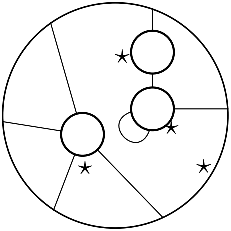
Question

Does there exist a non-trivial knot having the same Jones polynomial as the unknot?

Definition

A planar diagram has

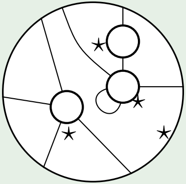
- *a finite number of inner boundary circles*
- *an outer boundary circle*
- *non-intersecting strings*
- *a marked point $\star$ on each boundary circle*



Definition (Jones)

A planar algebra is

- *a family of vector spaces* V_k, $k = 0, 1, 2, \ldots$, *and*
- *an interpretation of any planar diagram as a multi-linear map*

among V_i*:*  $: V_2 \times V_5 \times V_4 \to V_7$

together with some axioms ensuring that diagrams act consistently.

Example

Temperley-Lieb is a planar algebra, with planar diagrams acting by insertion and replacing-loops-by-δ.

Example

Let T_n be the vector space over $\mathbb{C}$ spanned by tangles of string with n fixed endpoints, up to (boundary-preserving) isotopy. The T_n form a planar algebra, with planar diagrams acting by insertion.

The Jones polynomial extends to a homomorphism of planar algebras between $\mathcal{T} = \{T_n\}$ and $\mathcal{TL} = \{TL_n\}$

The n-color theorems

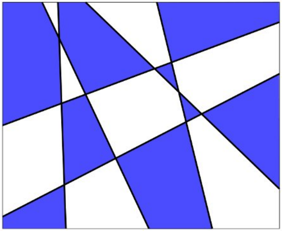
A graph can be n-colored if you can color its faces using n different colors such that adjacent regions are different colors. (Graphs with faces are embedded in a surface. We'll stick with planar graphs.)

Definition

The degree of a vertex is the number of edges it has coming into it.

The two-color theorem

Any planar graph where every vertex has even degree can be two-colored.



A three-color theorem (Grötzsch 1959)

Planar graphs with no degree-three vertices can be three-colored.

The five-color theorem (Heawood 1890, based on Kempe 1879)

Any planar graph can be five-colored.

A three-color theorem (Grötzsch 1959)

Planar graphs with no degree-three vertices can be three-colored.

The five-color theorem (Heawood 1890, based on Kempe 1879)

Any planar graph can be five-colored.

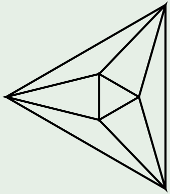
The four-color theorem (Appel-Haken 1976)

Any planar graph can be four-colored.

Definition/Theorem

The Euler characteristic of a graph is $V - E + F$. For planar graphs, $V - E + F = 2$.

Example



V=6

E=12

F=8

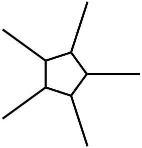
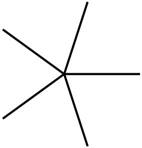
V-E+F=2

Corollary

Every planar graph has a face which is either a bigon, triangle, quadrilateral or pentagon.

A new proof of the four-color theorem:

First, observe:

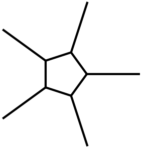
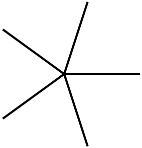
If I can color  then I can color . So,

replacing every degree-n vertex with a small n-gonal face doesn't change colorability.

Thus, if a coloring theorem is true for graphs where every vertex has degree three, it is true for all graphs.

A not-so-new proof of the five-color theorem:

First, observe:

If I can color  then I can color . So,

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Thus, if a coloring theorem is true for graphs where every vertex has degree three, it is true for all graphs.

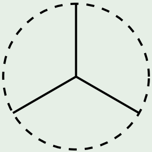
Definition

The color-counting planar algebra: The vector space V_k is functions from length-k sequences of colors to numbers:

$$V_k = \{f : \{\text{colors}\}^k \to \mathbb{R}\}$$

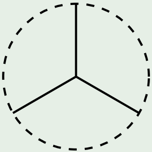
Any planar graph (with a boundary) is a function from a sequence of colors, to a number: how many ways are there to color in this graph so that the boundary colors are the given sequence?

Example

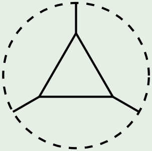


$$\{1,2,3\} \to 1$$
$$\{1,2,2\} \to 0$$
$$\{i,j,k\} \to \begin{cases} 1 & \text{if } i,j,k \text{ distinct;} \\ 0 & \text{else.} \end{cases}$$

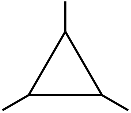
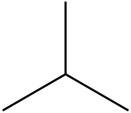
Example

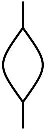
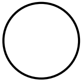


$$\{1,2,3\} \to 1$$
$$\{1,2,2\} \to 0$$
$$\{i,j,k\} \to \begin{cases} 1 & \text{if } i,j,k \text{ distinct;} \\ 0 & \text{else.} \end{cases}$$

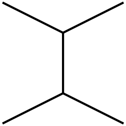
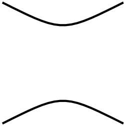
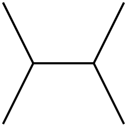
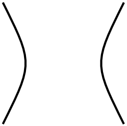


$$\{1,2,3\} \to n-3$$
$$\{1,2,2\} \to 0$$
$$\{i,j,k\} \to \begin{cases} n-3 & \text{if } i,j,k \text{ distinct;} \\ 0 & \text{else.} \end{cases}$$

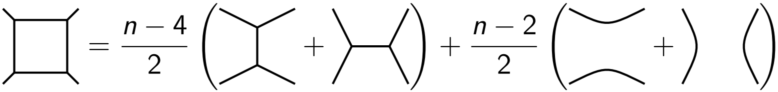
So,  $= (n-3)$  .

Similarly,  $= (n-2)$ and  $= (n-1)$.

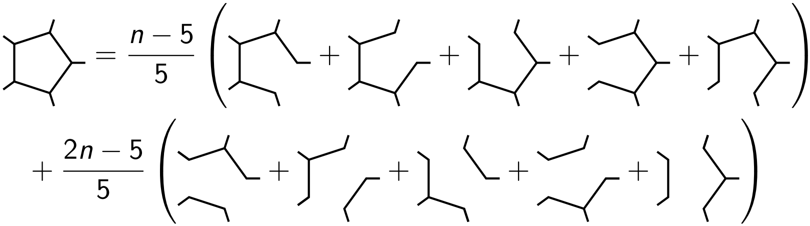
We also have a less obvious relation:

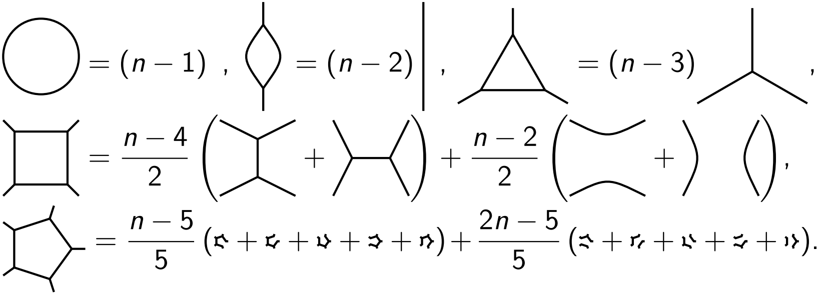
 $+$  $=$  $+$  .

This last relation can be used to prove two more relations:



and





All these face-removing relations are positive for $n \geq 5$.
Any planar graph contains at least one circle, bigon, triangle, quadrilateral or pentagon (via Euler characteristic). So apply one of these positive relations and repeat until you have nothing left but a sum of positive multiples of the empty diagram.

The standard invariant of a subfactor is a planar algebra $\mathcal{P}$ with some extra structure. Most significantly, $\mathcal{P}_0$ is one-dimensional and each $\mathcal{P}_k$ has an adjoint $*$ such that $\langle x, y \rangle := \operatorname{tr}(y^* x)$ is an inner product. Thus, our planar algebras have extra geometric structure.

A planar algebra with these properties a <u>subfactor planar algebra</u>.

Theorem (Jones, Popa)

Subfactors give subfactor planar algebras, and subfactor planar algebras give subfactors.

Example

Temperley-Lieb is a subfactor planar algebra if $\delta \geq 2$:

- TL_0 is one dimensional
- Positive definiteness is the difficulty, and where $\delta \geq 2$ comes in.

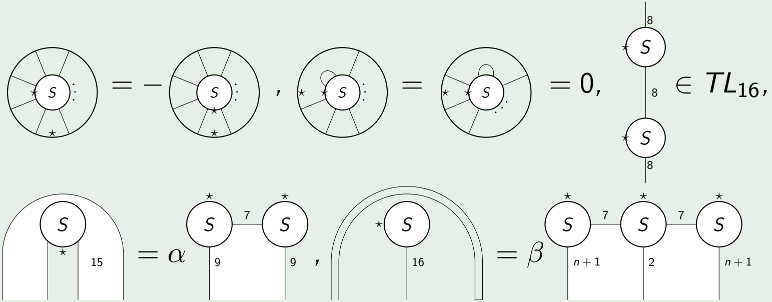
But wait, there's more!

Theorem (Jones)

Any subfactor planar algebra contains a copy of $\mathcal{TL}$ (if the index of the subfactor is four or more) or a quotient of $\mathcal{TL}$ (if the index is under 4).

Theorem (Bigelow, Morrison, P., Snyder)

The extended Haagerup planar algebra $\mathcal{H}$ *is the positive definite planar algebra generated by a single generator* S *with 16 strands, subject to the relations*



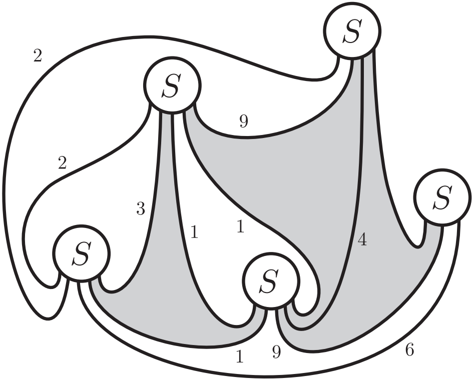
It is a (non-trivial) subfactor planar algebra.

Proof sketch: Any set of generators and relations give us a planar algebra; how do we know that $\mathcal{H}$ is a subfactor planar algebra? How do we know $\mathcal{H}$ isn't the trivial planar algebra?

Non-triviality follows from embedding $\mathcal{H}$ in a larger and easier planar algebra. We check that the image there is non-zero.

To see that $\mathcal{H}$ is a subfactor planar algebra, we need to show that $\dim(\mathcal{H}_0) = 1$. That is, how do we see that any closed diagram as a multiple of the empty diagram? We need to describe an 'evaluation algorithm' which will reduce any diagram to a multiple of the empty diagram .

For extended Haagerup, we treat each copy of S as a 'jellyfish' and use the substitute braiding relations to 'swim' each jellyfish to the top of the diagram. Begin with arbitrary closed diagram of Ss.



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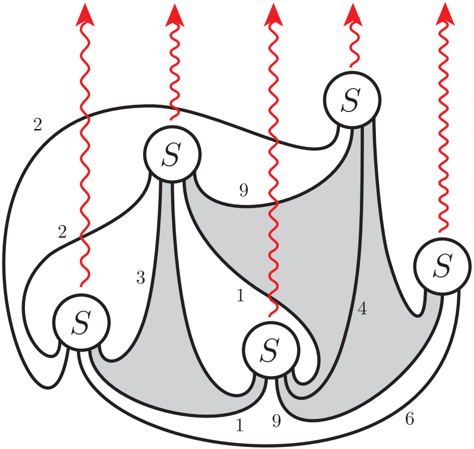

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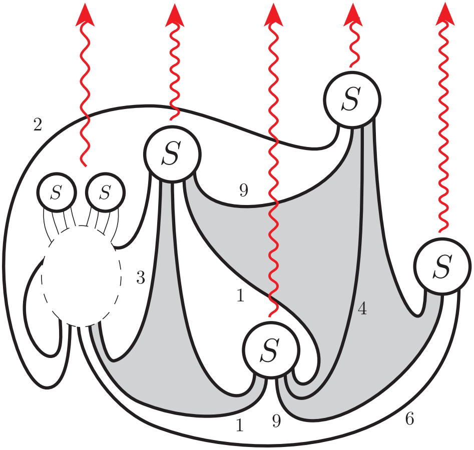

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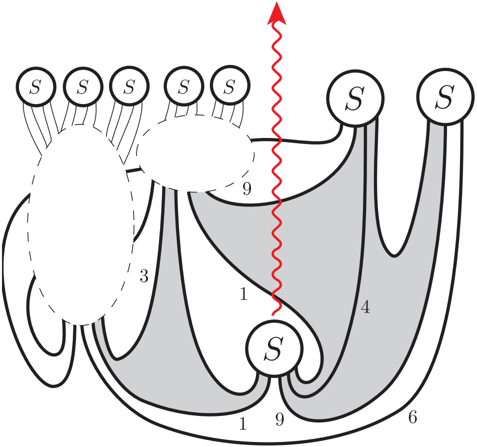

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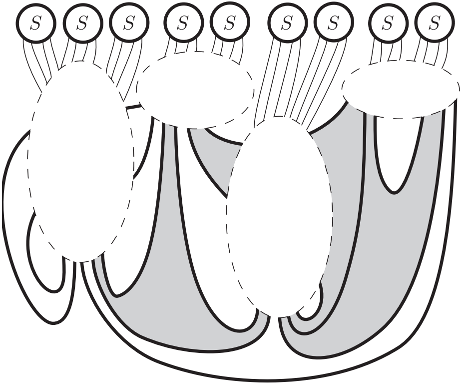
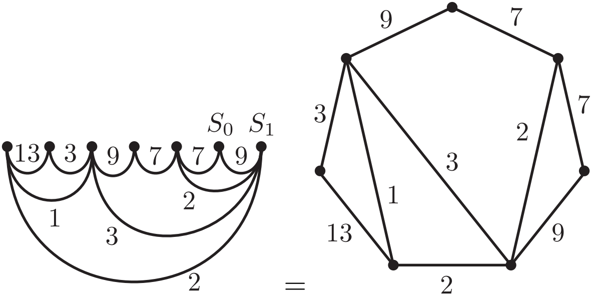


The diagram now looks like a polygon with some diagonals, labelled by the numbers of strands connecting generators.



- Each such polygon has a corner, and the generator there is connected to one of its neighbors by at least 8 edges.
- Use $S^2 \in TL$ to reduce the number of generators, and recursively evaluate the entire diagram.

The End!